ANNEX 78

**FINANCE MANAGEMENT**

(Working material)

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# 7 RISK REDUCTION BY DIVERSIFICATION

## 7.1 RANDOM VARIABLES

Frequently the amount of money to be obtained when selling an asset is uncertain at the time of purchase. In that case the return is random and can be described in probabilistic terms.

If quantity can obtain several outcomes, which are uncertain, we shall say that this quantity is **random variable**.

Random variables usually are denoted by capital letters $X$ , $Y$ , … .
Each outcome usually associates with the relative chance of an occurrence called probability.
Outcomes of random variables are denoted by corresponding small letters:
$X: \; x_1, x_2, ..., x_n$ ,

but corresponding probabilities by $p_1, p_2, ..., p_n$ where $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$ for all $i \in \{1, 2, ..., n\}$.

### DISCRETE RANDOM VARIABLES

> A random variable is **discrete** if it can take on no more than a countable number of values

Information can be ordered in table:

| $X$ | $x_1$ | $x_2$ | $\cdots$ | $x_n$ |
|---|---|---|---|---|
| $p$ | $p_1$ | $p_2$ | $\cdots$ | $p_n$ |

Then we are saying that probability distribution is given.

It is common to display the probabilities associated with random variable graphically as a density.
For example

**Expected value**

The expected value of a discrete random variable $X$ is just the average value obtained by multiplying values of random variable by corresponding probabilities and adding all together:

$$E(X) = \sum_{i=1}^{n} x_i \cdot p_i \, .$$

Often mean and mean value is used for the expected value.
Basic properties:

1. Certain value. If $Y$ has a certain values $y$, then $E(Y) = y$.
2. Linearity. If $Y$ and $Z$ are random, ten $E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y)$ for any real numbers $\alpha$ and $\beta$.
3. Nonnegativity. If $X$ is random but never less than zero, then $E(X) \geq 0$. This is sign-preserving property.

**Variance**

Variance is measure of dispersion and it is defined as

$$Var(X) = E((X - E(X))^2) \text{ or } Var(X) = E((X - \overline{x})^2)$$

There is a simple formula for variance that is useful in computations:

$$Var(X) = E(X^2) - (E(X))^2 \text{ or } Var(X) = E(X^2) - (\overline{x})^2$$

Basic properties:

1. Certain value. If $Y$ has a certain values $y$, then $Var(Y) = 0$.
2. Constant multiplayer. $Var(\alpha X) = \alpha^2 \cdot Var(X)$ for any real number $\alpha$.

We frequently use the square root of the variance, denoted by $\sigma$ and called the **standard deviation**:

$$\sigma_Y = \sqrt{Var(Y)} \, .$$

Example 7.1

Possible rates of return of asset A and related probabilities are shown in table:

| | | | |
|---|---|---|---|
| Possible rates of return (%) | 20 | 5 | -10 |
| Probabilities | 0,4 | 0,4 | 0,2 |

Calculate expected rate of return, variance and standard deviation of rates of return. Sketch the graph of probability function.

## CONTINUOUS RANDOM VARIABLES

A random variable is **continuous** if it can take any value in an interval

If the outcomes of variable can take any real value in an interval as, for example, temperature in a next day, a probability density function $f(x)$ describes the probability in following way:

$$P(a < X < b) = \int_a^b f(x) \cdot dx = S$$

The cumulative distribution function $F(x)$ of a continuous random variable $X$ expresses the probability that $X$ does not exceed the value $x$:

$$F(x) = P(X \leq x).$$

Expected value:

$$E(x) = \int_{-\infty}^{+\infty} x f(x) dx.$$

Variance:

$$Var(x) = \int_{-\infty}^{+\infty} x^2 f(x) dx - \left(E(x)\right)^2.$$

## THE NORMAL DISTRIBUTION

If the random variable X has probability density function

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2} \text{ for } -\infty < x < \infty$$

Where $\mu$ and $\sigma^2$ are any number such that $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$, then X is said to follow a normal distribution.

Some properties of the normal distribution:

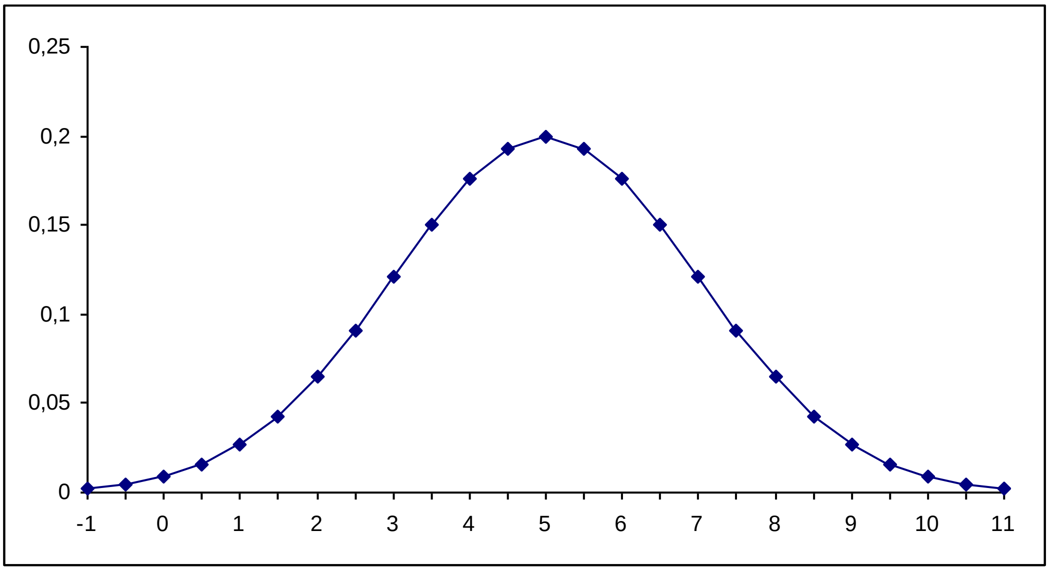
1. The mean of the random variable is $\mu$; that is

$$E(X) = \mu$$

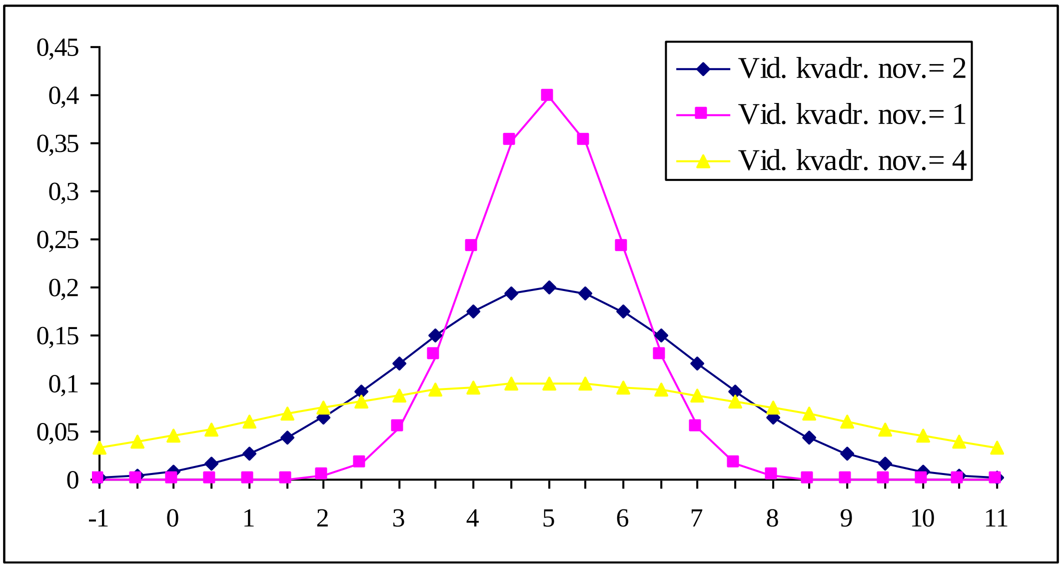
2. The variance of the random variable is $\sigma^2$; that is

$$Var(X) = E\left[(X - \mu_x)^2\right] = \sigma_x^2$$

3. The shape of the probability density function is a symmetric bell shaped curve centred on the mean $\mu$.




4. The height of curve and it inflection points are defined by $\sigma$.




Suppose that X is a normal random variable with mean $\mu$ and variance $\sigma^2$, then the cumulative distribution function $F(x)$ is

$$F(x) = P(X \le x)$$

Let X be a normal random variable with cumulative distribution function $F(x)$, and let a and b be two possible values of X, with a<b Then

$$P(a < x < b) = F(b) - F(a)$$

**Standard normal probability distribution**

Let Z be a normal random variable with mean 0 and variance 1 that is,

$$Z \sim N(0,1)$$

then Z is said to follow the standard normal distribution.

Example 7.2

Data have a normal distribution with mean 6 and standard deviation 2. Calculate probabilities

1) $P(2 < X < 4)$ 2) $P(X < 7)$ 3) $P(X > 6,5)$ 4) $P(4 < X < 8)$ 5) $P(2 < X < 10)$ 6) $P(0 < X < 12)$

## THE UNIFORM DISTRIBUTION

The random variable $X$ is ***uniformly distributed*** in closed interval $[a,b]$, if its density function is constant in that interval $[a,b]$.

***The density function:***

$$f(x)=\begin{cases}0, & if \quad x<a \\ \dfrac{1}{b-a}, & if \quad x\in[a,b] \\ 0, & if \quad x>b\end{cases}.$$

***The cumulative distribution function:***

$$F(x)=\begin{cases}0, & if \quad x<a \\ \dfrac{x-a}{b-a}, & if \quad x\in[a,b] \\ 1, & if \quad x>b\end{cases}.$$

***Characteristics of uniformly distributed random variable:***

$$E(X)=\frac{a+b}{2}, \; D(X)=\sigma_X^2=\frac{(b-a)^2}{12}, \; \sigma_X=\frac{b-a}{\sqrt{12}}$$

Example 7.3

Claim arrival in insurance company after their happening is uniformly distributed between 2 and 10 days.

1. Write formula and sketch a graph of density function.
2. Write formula and sketch a graph of cumulative distribution function.
3. Calculate average number of days passed from moment when accident has happened till it is reported in insurance company and its standard deviation.
4. Calculate probability that the next claim will be reported not early than 7 days.
5. Calculate probability that the next claim will be reported early than in 4 days.

## THE EXPONENTIAL DISTRIBUTION

Random variable $X$ has an ***exponential distribution*** if the random variable $X$ cannot take negative values and has probability density function is

$$f_X(x)=\frac{e^{-x/\mu}}{\mu} \text{ for } x\geq 0.$$

where $\mu$ is any positive number.

The cumulative distribution function is $F_X(X)=1-e^{-x/\mu}$

The distribution has mean $\mu$ and variance $\mu^2$

Example 7.4

Service of customers at a bank information desk follows an exponential distribution, with mean service time 4 minutes. What is the probability that a customer service will take longer than 9 minutes?

Example 7.5

Claim reporting time in accident insurance follows an exponential distribution with mean 10 days.

1) Write formula for density function and sketch its graph.
2) Write formula for distribution function and sketch its graph.
3) Calculate standard deviation.
4) Find probability than one particular accident will be reported no earlier than in 20 days.
5) Find probability than one particular accident will be reported in 5 days.

The exponential distribution related to the Poisson distribution. If number of occurrences of an event in time interval follows a Poisson distribution with mean $\lambda$, then the time between successive occurrences of the event follows an exponential distribution

## SEVERAL RANDOM VARIABLES

If we are interested in several random variables all possible outcomes have to be investigated.

Two random variables can be

**independent**
Two random variables are said to be independent if the outcome probabilities for one variable do not depend on the outcome of the other.

**dependent**
Two random variables are said to be dependent if the outcome probabilities for one variable depend on the outcome of the other.

Let $X$ and $Y$ be a pair of random variables, with respective means $\mu_X$ and $\mu_Y$. The expected value of $(X-\mu_X)\cdot(Y-\mu_Y)$ is called the covariance between $X$ and $Y$.

That is

$$Cov(X,Y)=E\left[(X-\mu_X)\cdot(Y-\mu_Y)\right]$$

or alternative expression is

$$Cov(X,Y)=E(X\cdot Y)-\mu_X\cdot\mu_Y.$$

If the random variables $X$ and $Y$ are independent, then the covariance between them is 0.

Covariance for sample is calculated in following way:

$$Cov(X,Y)=\frac{\sum_{i=1}^{n}(x_i-\overline{x})\cdot(y_i-\overline{y})}{n-1} \quad \text{jeb} \quad Cov\left(X,Y\right)=\frac{n\cdot\sum_{i=1}^{n}x_i\cdot y_i-\left(\sum_{i=1}^{n}x_i\right)\cdot\left(\sum_{i=1}^{n}y_i\right)}{n\cdot(n-1)}$$

Example 7.6

The price of goods and sold amount are shown in the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Amount sold | 12 | 18 | 29 | 16 | 25 | 35 | 15 | 11 |
| Price (LVL) | 19,5 | 20,5 | 19,0 | 21,5 | 20,5 | 20,0 | 18,0 | 17,0 |

Calculate covariance and sketch the dot diagram.

Coefficient of correlation

for population

$$\rho=\frac{Cov(X,Y)}{\sigma_X\cdot\sigma_Y}=\frac{\sum_{i=1}^{N}(x_i-\mu_x)(y_i-\mu_y)}{\sqrt{(\sum_{i=1}^{N}x_i^2-n\mu_x{}^2)(\sum_{i=1}^{N}y_i^2-n\mu_y{}^2)}}$$

for sample

$$r = \frac{Cov(X,Y)}{s_X \cdot s_y} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2)(\sum_{i=1}^{n} y_i^2 - n\bar{y}^2)}}$$

$$-1 \le r \le 1$$

Example 7.7

Calculate coefficient of correlation or data given in example 7.2.

## 7.2 MEAN-VARIANCE PORTFOLIO MODEL (The Markowitz Model)

**Main assumptions:**

1. Investor takes into account only two parameters: security mean return and return variance to choose securites for portfolio.

2. Investor has squared utility function.

3. It is assumed that return of security is normally or lognormally distributed.

4. It is assumed that
   a) Investor is nonsatiated,
      Investor with given risk will choose portfolio wit higest possible return.
   b) Investor does not like risk (risk averse).
      Investor with giver rate of return will choose portfolio with smalest possible risk.

**Used notations:**

$r_i$ - rate of return of $i$ security,

$\sigma_i^2$ - variance of return of $i$ security ( $\sigma_i$ - corresponding standard deviation),

$E(R_i) = \bar{r}_i$ - expected rate of return of $i$ security,

$\alpha_i$ - amount of $i$ security in portfolio- ( $\sum_{i=1}^{n} \alpha_i = 1$ ),

$r_P$ -portfolio rate of return,

$\sigma_P^2$ - variance of portfolio rate of return ( $\sigma_p$ - standard deviation),

$E(r_P) = \overline{r_P}$ - portfolio expected rate of return,

$\sigma_{i,k}$ - covariance between rates of return of $i$ and $k$ securities,

$\rho_{i,k}$ - corresponding coefficient of correlation.

**MODEL FORMULAS**

$$\overline{r_P} = \sum_{i=1}^{n} \alpha_i \cdot \overline{r_i} \text{ , kur } \sum_{i=1}^{n} \alpha_i = 1$$

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_i \cdot \alpha_k \cdot \sigma_{i,k}$$

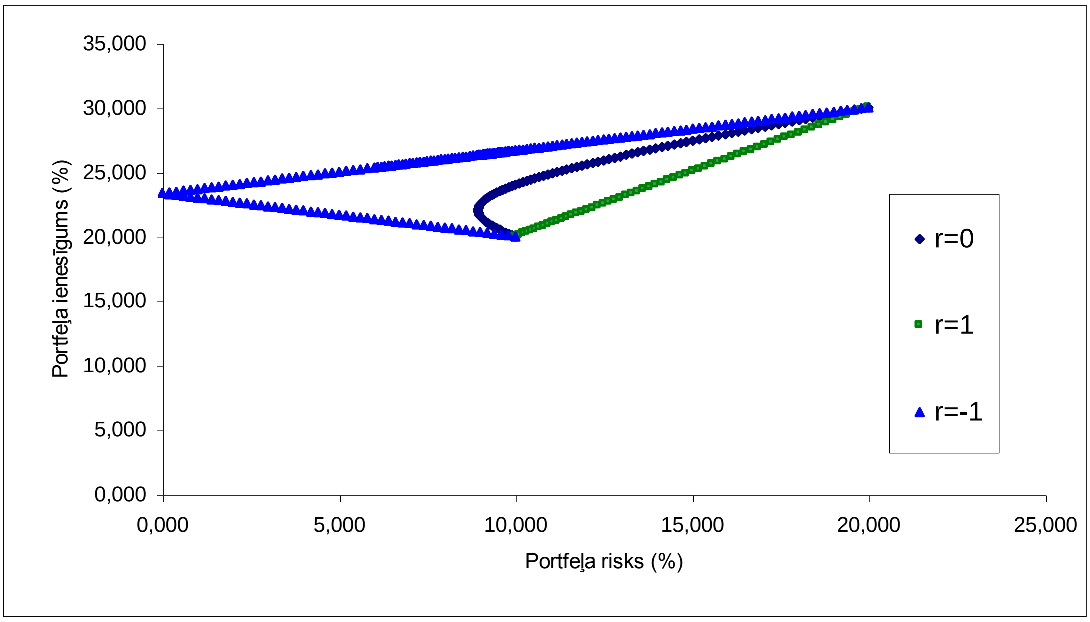
**CASE OF TWO SECURITIES**

$$\overline{r_p} = \alpha \cdot \overline{r_1} + (1-\alpha) \cdot \overline{r_2}$$

$$\sigma_p^2 = \alpha^2 \cdot \sigma_1^2 + 2 \cdot \rho_{1,2} \cdot \alpha \cdot (1-\alpha) \cdot \sigma_1 \cdot \sigma_2 + (1-\alpha)^2 \cdot \sigma_2^2$$

Example 7.8

Corresponding historical rate of return of stock A and B are shown in given table. It is forecasted that expected rate of return of stock A is 20% with standard deviation 10% and it is forecasted that expected rate of return of stock B is 30% with standard deviation 20%. Calculate how much is needed to buy each security to have portfolio with smalest risk. Calculate in this case rate or return of portfolio and its standard deviation. Sketch the graph to show relationship between rate of return of portfolio and its risk.




The set of points that correspond to portfolios is called the feasible set or feasible region.

Portfolio is called effective if

1) With given risk is not possible to find better portfolio (with larger rate of return),
2) With given rate of return is not possible to find better portfolio (with smaller risk)

Set of effective portfolios are called efficient frontier.

# 8 PENSION CALCULATIONS

## 8.1 MORTALITY TABLES AND THEIR FUNCTIONS

Table:

| Age<br>x | Number of people alive<br>in age x | Number of people died in<br>age x |
|---|---|---|
| 0 | $l_0$ | $d_0$ |
| 1 | $l_1$ | $d_1$ |
| 2 | $l_2$ | $d_2$ |
| 3 | $l_3$ | $d_3$ |
| ... | ... | ... |

where

$x$ – age in whole years,

$l_x$ – number of people alive in age $x$,

$d_x$ – number of people alive in age $x$.

***Main assumption:***

Group of people is closed.

Unique reason for decreasing of number of alive people is death.

Relationship:

$$l_{x+1} = l_x - d_x$$

*Example 8.1* Calculate number of death in each age.

***Functions of mortality table.***

1) More simply functions:

$p_x$ - probability that people who is alive at age x will be alive after one year ,

${}_t p_x$ - probability that people who is alive at age x will be alive after $t$ year,

$q_x$- probability that people who is alive in age *x*, will die during next year,

${}_t q_x$ - probability that people who is alive in age *x*, will die during next *t* year.

Relations between functions:

$$p_x = \frac{l_{x+1}}{l_x} \qquad q_x = \frac{d_x}{l_x} \quad \text{or} \quad q_x = 1 - p_x$$

$${}_t p_x = \frac{l_{x+t}}{l_x} \qquad {}_t q_x = \frac{l_x - l_{x+t}}{l_x} \quad \text{or} \quad {}_t q_x = 1 - {}_t p_x$$

$${}_{m/u} p_x = \frac{l_{x+m+u}}{l_x} \quad \text{and} \quad {}_{m/u} q_x = \frac{l_{x+m} - l_{x+m+u}}{l_x}.$$

*Example 8.2* Let's read, write formulas and calculate given functions:

$p_{30}$, ${}_5 p_{45}$, $q_{40}$, ${}_3 q_{45}$, ${}_{10} p_{30}$, $q_{40}$, ${}_{10} q_{50}$ ${}_{3/} q_{42}$, ${}_{3/10} q_{65}$, ${}_{1/10} q_{62}$, ${}_{25/11} q_{23}$.

*Example 8.3* Write with symbols, formulas and calculate probability that:

a) 30 years old man will die not reaching 45 years,
b) 25 years old female will be still alive in 40 years,
c) 38 years old female will die during next 17,
d) 55 years old man will die betwenn 55 and 70 years,
e) 28 years old man will die betwee 62 and 70 years,
f) 45 years old female will die between 65 and 75 years.

***Expected length of life in discrete case (in whole years)***

$$P(K_x = k) = P(k \le T_x < k+1) = {}_k p_x \cdot q_{x+k}$$

*Example 8.4* Read, write calculation formulas and calculate:

$P(K_{50} = 10)$, $P(K_{45} = 5)$, $P(K_{65} = 15)$.

Expected length of life $e_x = E(K_x) = \sum_{k=1}^{[\omega-x]} {}_k p_x$

*Example 8.5* Calculate:

1) 20 years old man expected length of life,
2) 62 years old man expected length of life.

## 8.2 LIFE ANNUITIES

1) *Life anuity when payments are in the end of each period:*

One unit is paid to person who has x years to start from that age all life while the person is alive.

$$a_x = E(a_{\overline{K|}}) \quad , \quad \text{because } Z = a_{\overline{K|}}$$

Table

| | | | |
|---|---|---|---|
| $Z$ | $a_{\overline{1\|}}$ | $a_{\overline{2\|}}$ | $\cdots$ |
| $P$ | ${}_1 p_x \cdot {}_{1\|}q_x$ | ${}_2 p_x \cdot {}_{2\|}q_x$ | $\cdots$ |

$$a_x = \sum_{k=1}^{[\omega-x]} v^k \cdot \frac{l_{x+k}}{l_x}$$

*Example 8.6* Calculate how large yearly pension will receive 63 years old man while he will be alive , if it starts from next year and his accumulated value is 120 000 EUR, assuming 3% guaranteed annual effective interest rate.

2) *Life anuity when payments are in the beginning of each period:*

One unit is paid to person who has *x* years to start from that age in the beginning of period all life while the person is alive.

$$\ddot{a}_x = E(a_{\overline{K+1|}}) \quad , \quad \text{jo } Z = \ddot{a}_{\overline{K+1|}}$$

$$\ddot{a}_x = \sum_{k=0}^{[\omega-x]} v^k \cdot \frac{l_{x+k}}{l_x}$$

.

*Example 8.7* Calculate how large has to be accumulation for 65 years old man if he wish to receive 6 000 EUR in each year to start from 65 years while he will be alive, assuming 2% guaranteed annual effective interest rate.

3) Term life anuities:

Life annuities during concrete term *n*.

If payments are in the end of period:

$$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \cdot \frac{l_{x+k}}{l_x}$$

If payments are in the beginning of period:

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \cdot \frac{l_{x+k}}{l_x}$$

Example 8.8 Calculate how large has to be accumulated value to female in the age 55 to start to receive from that age 600 EUR life pension during the next 10 years, assuming 1.5% guaranteed annual effective rate of interest.

4) Deferred life anuities:

If annuities are deferred for *m* periods and if they started only if person is alive, then:

$${}_{m/}a_x = v^m \cdot {}_m p_x \cdot a_{x+m} \qquad {}_{m/}\ddot{a}_x = v^m \cdot {}_m p_x \cdot \ddot{a}_{x+m}$$

$${}_{m/}a_{x:\overline{n}|} = v^m \cdot {}_m p_x \cdot a_{x+m:\overline{n}|} \qquad {}_{m/}\ddot{a}_{x:\overline{n}|} = v^m \cdot {}_m p_x \cdot \ddot{a}_{x+m:\overline{n}|}$$

Example 8.9 Calculate how large life pension will receive 45 years old female to start from 65 years if she has started to pay for accumulation 600 EUR in year to start from 45 years, assuming 2% guaranteed annual effective interest rate.

5) Guaranteed life annuities:

$$a_{\overline{\overline{x:n}|}} = a_{\overline{n}|} + v^n \cdot \frac{l_{x+n}}{l_x} \cdot a_{x+n} \qquad \text{or} \qquad \ddot{a}_{\overline{\overline{x:n}|}} = \ddot{a}_{\overline{n}|} + v^n \cdot \frac{l_{x+n}}{l_x} \cdot \ddot{a}_{x+n}$$

*Example 8.10* Calculate EUR 1000 yearly life pension, which we have to start to pay 55 years old female with guaranteed period 10 years in the beginning of each year, present value, assuming 2.5% guaranteed annual effective interest rate.

## 8.3 CALCULATIONS WITH EXPENSES

Expenses:

Commissions – initial

from further payment

initial expenses – policy documentation preparing

further expenses – administration of systems

data system administration

actuary job

claim expenses – document checking

bank transfers

$$\text{PV of in go} = \text{PV of outgo}$$

*Example 8.11* Calculate how large pension will be in example 8 described situation if it is assumed 5% expenses from each payment in and 0,1% from each pension.

## 8.4 RESERVING

Definition:

| |
|---|
| Reserve is fond which has to be kept in time t for each alive pensioner in the end of t year to be able to cover liability to pay pension. |

**Retrospective and prospective reserve:**

Retro ${}_tV^{retro} = Accumulated\ \ past\ \ premiums - acumulated\ \ past\ \ payments\ and\ \exp ensis$

$\Pr ospective\ {}_tV^{pro} = PV\ \ of\ \ \ future\ outpayments - PV\ of\ \ \ future\ income$

*Example 8.12* Calculate all prospective reserves in example 11 described situation.

## 8.5 EXERCISES

1. 50 years old man wish to buy life pension to start from 65 years 600 EUR each year by paying premium once per year in the beginning of year to start from 50 years to 64 years age (including).
    a) Calculate how many he has to pay if there is assuming 10% expenses from each premium paid and 0.2% from each pension.
    b) Calculate all reserves.
2. Female in age 30 years wish to but term life pension 900 EUR in the beginning of each year to start from her 60 to 80 (not including) years. She is ready pai premium 20 years to start from 30 years age in the beginning of of each year.
    1. Calculate amount of premium if there are assumed expensis 9% from each premium, 10 EUR initial expensis(together with first premium) and 0.3% from every pension. It is assumed 1.5% guaranteed interest rat per anum.
    2. Calculate at least one reserve in each payment period.