# MACROECONOMETRIC MODELS II

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# Q & A

# CONTENTS

# SINGLE-EQUATION ECONOMETRICS–A REMINDER

Summary

# A SIMPLE EXAMPLE: NATIONAL ECONOMY

| | |
|---|---|
| Definition Y: Gross Domestic Product | $Y(t) = C(t) + I(t) + G(t) + X(t) - M\ (t)$ |
| Consumption function of private households C | $C(t) = \alpha 0 + \alpha 1 * Y(t-1)$ |
| Private Investment function I | $I(t) = \ \lambda\ * (Y(t) - Y(t-1))$ |
| Government spending G | G (t): exogenous |
| Exports X | X(t): exogenous |
| Imports M | $M(t) = \gamma 0 + y1 * Y(t)$ |

# SUMMARY OF SINGLE EQUATION

| | |
|---|---|
| Theoretical Model | $\mathbf{Y = \beta 1 + \beta 2 * X + u}$ |
| Statistical model: Fitted values | $\hat{Y} = b_1 + b_2 X$ |
| OLS Estimator of b1 (Intercept) | $b_1 = \bar{Y} - b_2 \bar{X}$ |
| OLS Estimator of b2 (slope) | $b_2 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$ |
| Goodness of fit: | $R^2 = \frac{ESS}{TSS} = \frac{\sum (\hat{Y}_i - \bar{Y})^2}{\sum (Y_i - \bar{Y})^2}$ |

# MODEL SOFTWARE: *gretl*

- **gretl: Gnu Regression, Econometrics and Time-series Library**

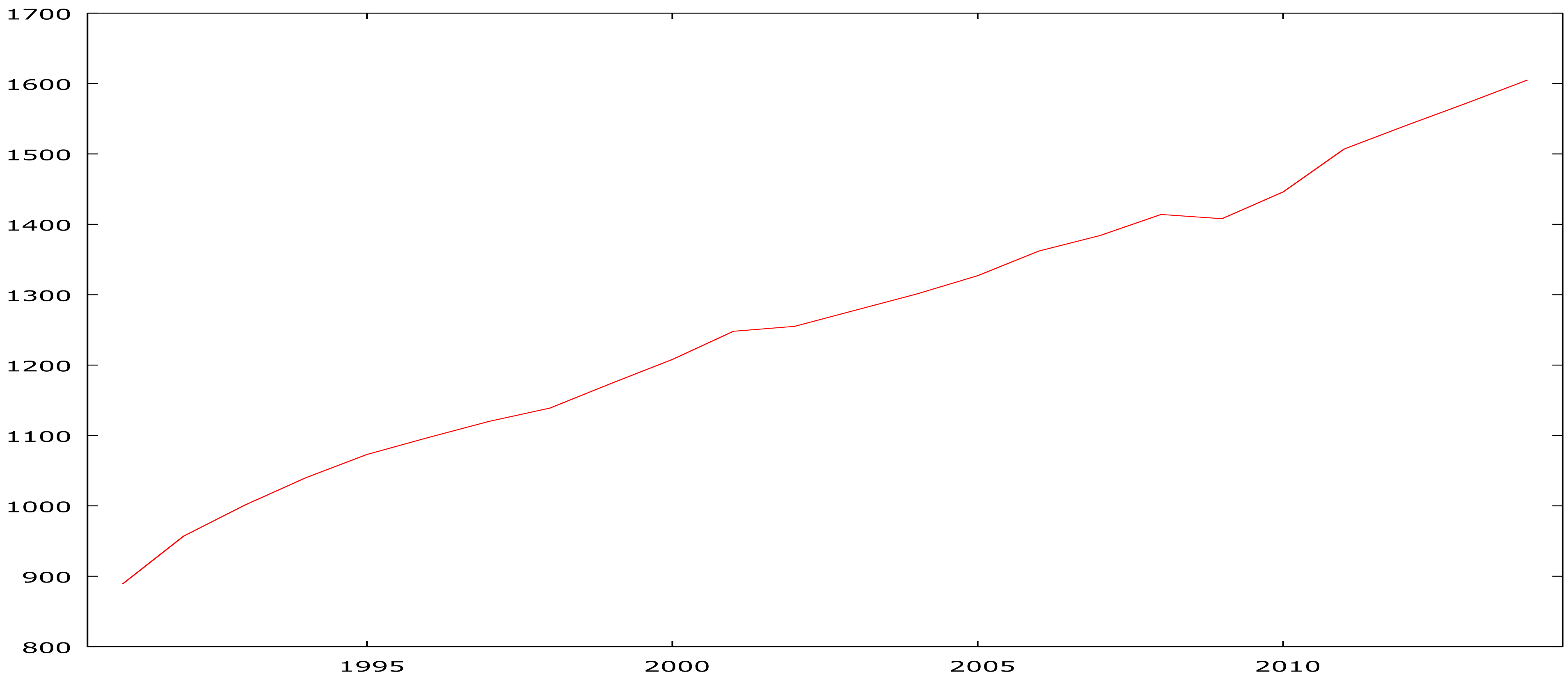
- Source in Internet for a free download
- http://gretl.sourceforge.net/

# SINGLE EQUATION ESTIMATION

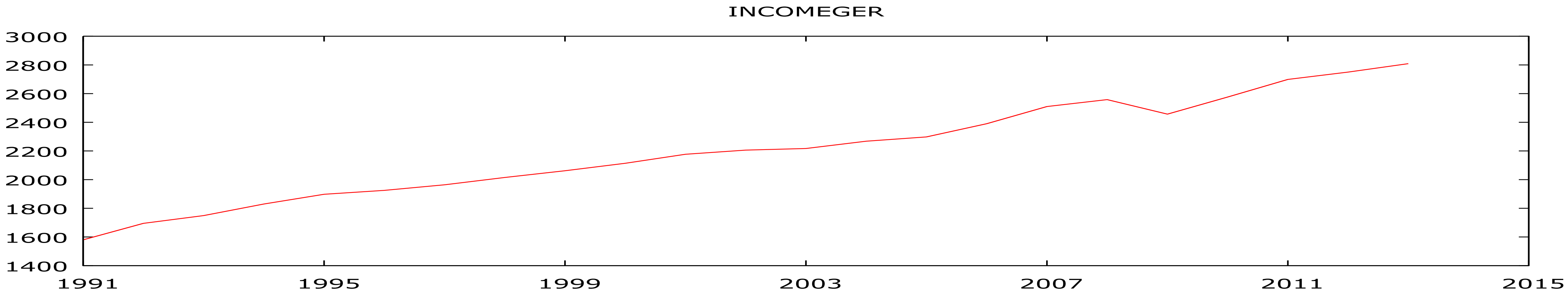
- Application of "*gretl*" econometric software

- **Consumption function**
- **Investment function**
- **Import function**

# Consumption, Germany Time Series



# GDP, Germany, Time Series



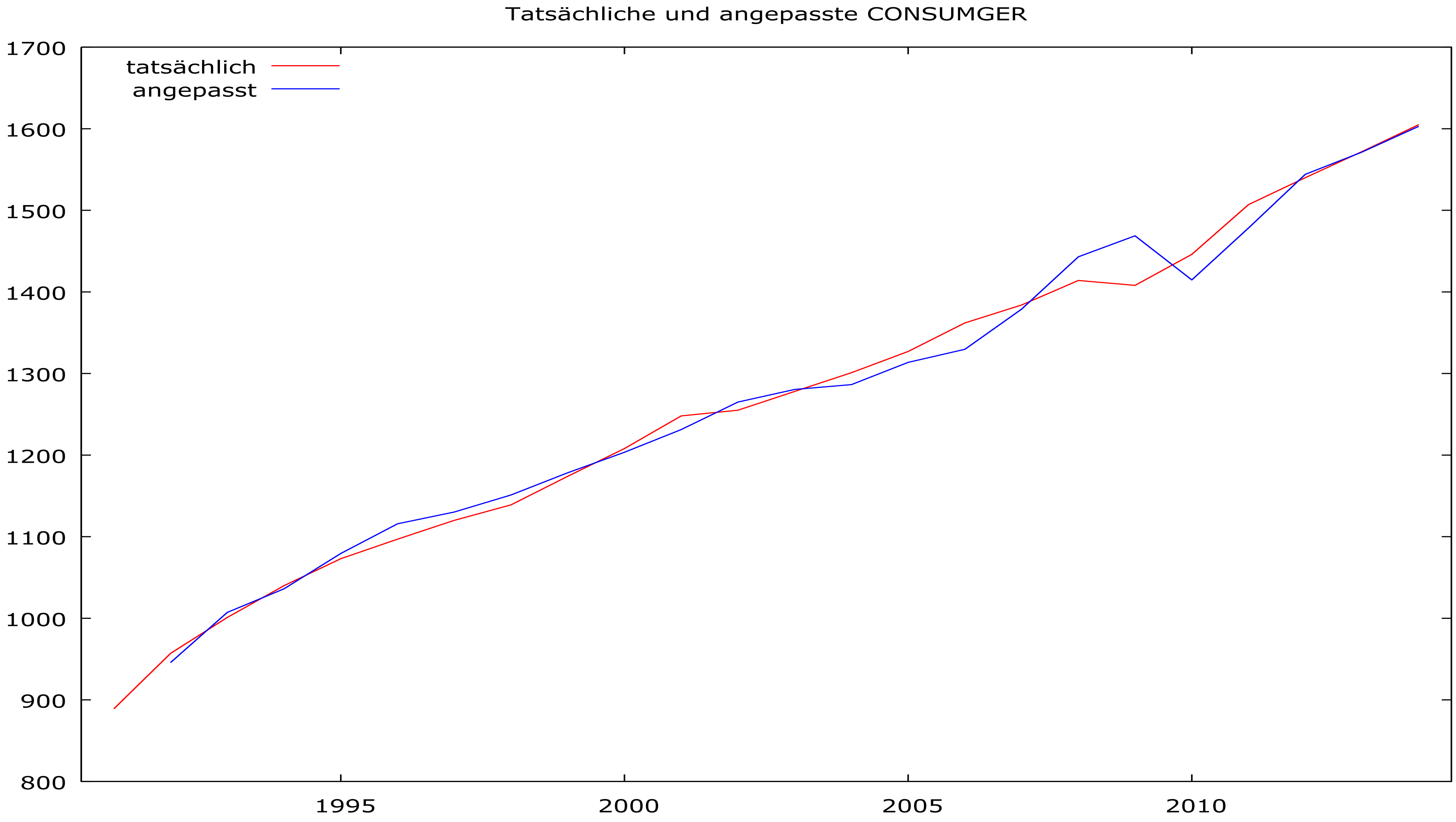
# Consumption Function

- Modell 1: KQ, benutze die Beobachtungen 1992-2014 (T = 23)
- Abhängige Variable: CONSUMGER

| | ***Koeffizient*** | *Std. Fehler* | ***t-Quotient*** | *p-Wert* | |
|---|---|---|---|---|---|
| const | 100,847 | 28,4018 | 3,5507 | 0,0019 | *** |
| INCOMEGER_1 | 0,534741 | 0,0127219 | 42,0333 | <0,0001 | *** |

- Mittel d. abh. Var. 1280,696 Stdabw. d. abh. Var. 187,2959
- Summe d. quad. Res. 9065,268 Stdfehler d. Regress. 20,77690
- **R-Quadrat 0,988254** **Korrigiertes R-Quadrat 0,987694**
- **F(1, 21) 1766,796** P-Wert(F) 9,37e-22
- Log-Likelihood −101,3678 Akaike-Kriterium 206,7355
- Schwarz-Kriterium 209,0065 Hannan-Quinn-Kriterium 207,3067
- rho 0,169094 Durbin-Watson-Stat 1,647511

# Goodness of fit



# MULTI-EQUATION ECONOMETRICS: METHODS AND PROBLEMS

# MODEL ESTIMATION

- Application of "*gretl*" econometric software

- BEHAVIOURAL EQUATIONS
- **Consumption function**
- **Investment function**
- **Import function**
- **Government consumption**

- EXOGENOUS VARIABLES
- **Government consumption**
- **Exports**

- DEFINITION
- Gross domestic product

# Model Results I

```
gretl Output fÃ¼r MS 2015-09-01 03:39, Seite 1

Gleichungssystem, Scheinbar unverbundene Regressionen

Gleichung 1: SUR, benutze die Beobachtungen 1992-2013 (T = 22)
Abhängige Variable: CONSUMGER

                Koeffizient   Std.-fehler   t-Quotient    p-Wert
  -----------------------------------------------------------------
  INCOMEGER      0,564301     0,00155185      363,6      2,21e-041 ***

Mittel d. abh. Var.     1265,955   Stdabw. d. abh. Var.    177,5228
Summe d. quad. Res.     6605,304   Stdfehler d. Regress.   17,32747

Gleichung 2: SUR, benutze die Beobachtungen 1992-2013 (T = 22)
Abhängige Variable: INVESTGER
```

# Model Results II

```
                 Koeffizient    Std.-fehler    t-Quotient     p-Wert
  ----------------------------------------------------------------
  d_INCOMEGER      5,33774       0,830885        6,424       2,29e-06 ***

Mittel d. abh. Var.      471,6364   Stdabw. d. abh. Var.    42,33151
Summe d. quad. Res.       1892414   Stdfehler d. Regress.   293,2897

Gleichung 3: SUR, benutze die Beobachtungen 1992-2013 (T = 22)
Abhängige Variable: IMPORTGER

               Koeffizient    Std.-fehler    t-Quotient     p-Wert
  ---------------------------------------------------------------
  INCOMEGER     0,323789       0,0136547       23,71       1,21e-016 ***

Mittel d. abh. Var.      694,0455   Stdabw. d. abh. Var.    250,0664
Summe d. quad. Res.      464640,7   Stdfehler d. Regress.   145,3273
```

# Model Results III

```
Gleichungsübergreifende Residuen-Var-Kov-Mat
(Über der Diagonalen Korrelationen)

       300,24        (0,492)      (-0,558)
       2498,2        86019,       (-0,151)
      -1404,5       -6422,5        21120,

log-Determinante = 26,346
Breusch-Pagan-Test für diagonale Kovarianzmatrix:
  Chi-Quadrat(3) = 12,6597 [0,0054]
```

# SIMULTANEOUS EQUATION BIAS

Cf: Annex:

Presentation based on

**Dougherty: Introduction to Econometrics 4e**

http://global.oup.com/uk/orc/busecon/economics/dougherty4e/

# Q & A

# ANNEX

# SIMULTANEOUS EQUATION BIAS

Presentation based on

**Dougherty: Introduction to Econometrics 4e**

http://global.oup.com/uk/orc/busecon/economics/dougherty4e/

# SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

**This sequence shows why OLS is likely to yield inconsistent estimates in models composed of two or more simultaneous relationships.**

# SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad\qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

**In this example we suppose that we have data on *p*, the annual rate of price inflation, *w*, the annual rate of wage inflation, and *U*, the rate of unemployment, for a sample of countries.**

# SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

We hypothesize that increases in wages lead to increases in prices and so *p* is positively influenced by *w* ($\beta_2 > 0$).

# SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

We also suppose that workers try to protect their real wages by negotiating for increases in wages as prices rise, but their ability to so is the weaker, the greater is the rate of unemployment ($\alpha_2 > 0$, $\alpha_3 < 0$).

# SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

**endogenous:** ***p*, *w***

**The model involves some circularity, in that *w* is a determinant of *p,* and *p* is a determinant of *w*. Variables whose values are determined interactively within the model are described as endogenous variables.**

# SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

endogenous: *p*, *w*

exogenous: *U*

We will cut through the circularity by expressing *p* and *w* in terms of their ultimate determinants, *U* and the disturbance terms $u_p$ and $u_w$. Variables such as *U* whose values are determined outside the model are described as exogenous variables.

## SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2 (\alpha_1 + \alpha_2 p + \alpha_3 U + u_w) + u_p$$

We will start with *p*.  The first step is to substitute for *w* from the second equation.

# SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad\qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2 \left( \alpha_1 + \alpha_2 p + \alpha_3 U + u_w \right) + u_p$$

$$\left( 1 - \alpha_2 \beta_2 \right) p = \beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w$$

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

**We bring the terms involving *p* together on the left side of the equation and thus express *p* in terms of *U*, $u_p$, and $u_w$.**

# SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad\qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2 \left( \alpha_1 + \alpha_2 p + \alpha_3 U + u_w \right) + u_p$$

$$\left( 1 - \alpha_2 \beta_2 \right) p = \beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w$$

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

$$w = \alpha_1 + \alpha_2 \left( \beta_1 + \beta_2 w + u_p \right) + \alpha_3 U + u_w$$

Next we take the equation for *w* and substitute for *p* from the first equation.

# SIMULTANEOUS EQUATIONS BIAS

$$p = \beta_1 + \beta_2 w + u_p \qquad\qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2 (\alpha_1 + \alpha_2 p + \alpha_3 U + u_w) + u_p$$

$$(1 - \alpha_2 \beta_2) p = \beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w$$

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

$$w = \alpha_1 + \alpha_2 (\beta_1 + \beta_2 w + u_p) + \alpha_3 U + u_w$$

$$(1 - \alpha_2 \beta_2) w = \alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w$$

$$w = \frac{\alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2}$$

**We bring the terms involving *w* together on the left side of the equation and thus express *w* in terms of *U*, $u_p$, and $u_w$.**

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p$$

$$w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2 (\alpha_1 + \alpha_2 p + \alpha_3 U + u_w) + u_p$$

$$(1 - \alpha_2 \beta_2) p = \beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w$$

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

$$w = \alpha_1 + \alpha_2 (\beta_1 + \beta_2 w + u_p) + \alpha_3 U + u_w$$

$$(1 - \alpha_2 \beta_2) w = \alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w$$

$$w = \frac{\alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2}$$

The original equations, representing the economic relationships among the variables, are described as the structural equations.

# SIMULTANEOUS EQUATIONS BIAS

**structural equations**

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2 (\alpha_1 + \alpha_2 p + \alpha_3 U + u_w) + u_p$$

$$(1 - \alpha_2 \beta_2) p = \beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w$$

**reduced form equation**

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

$$w = \alpha_1 + \alpha_2 (\beta_1 + \beta_2 w + u_p) + \alpha_3 U + u_w$$

$$(1 - \alpha_2 \beta_2) w = \alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w$$

**reduced form equation**

$$w = \frac{\alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2}$$

**The equations expressing the endogenous variables in terms of the exogenous variable(s) and the disturbance terms are described as the reduced form equations.**

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2(\alpha_1 + \alpha_2 p + \alpha_3 U + u_w) + u_p$$

$$(1 - \alpha_2\beta_2)p = \beta_1 + \alpha_1\beta_2 + \alpha_3\beta_2 U + u_p + \beta_2 u_w$$

reduced form equation

$$p = \frac{\beta_1 + \alpha_1\beta_2 + \alpha_3\beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2\beta_2}$$

$$w = \alpha_1 + \alpha_2(\beta_1 + \beta_2 w + u_p) + \alpha_3 U + u_w$$

$$(1 - \alpha_2\beta_2)w = \alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2}$$

The reduced form equations have two important roles. They can indicate that we have a serious econometric problem, but they may also provide a solution to it.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2(\alpha_1 + \alpha_2 p + \alpha_3 U + u_w) + u_p$$

$$(1 - \alpha_2\beta_2)p = \beta_1 + \alpha_1\beta_2 + \alpha_3\beta_2 U + u_p + \beta_2 u_w$$

reduced form equation

$$p = \frac{\beta_1 + \alpha_1\beta_2 + \alpha_3\beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2\beta_2}$$

$$w = \alpha_1 + \alpha_2(\beta_1 + \beta_2 w + u_p) + \alpha_3 U + u_w$$

$$(1 - \alpha_2\beta_2)w = \alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2}$$

**The problem is the violation of Assumption B.7 that the disturbance term be distributed independently of the explanatory variable(s). In the first equation, *w* has a component $u_p$. OLS would therefore yield inconsistent estimates if used to fit the equation.**

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2(\alpha_1 + \alpha_2 p + \alpha_3 U + u_w) + u_p$$

$$(1 - \alpha_2\beta_2)p = \beta_1 + \alpha_1\beta_2 + \alpha_3\beta_2 U + u_p + \beta_2 u_w$$

reduced form equation

$$p = \frac{\beta_1 + \alpha_1\beta_2 + \alpha_3\beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2\beta_2}$$

$$w = \alpha_1 + \alpha_2(\beta_1 + \beta_2 w + u_p) + \alpha_3 U + u_w$$

$$(1 - \alpha_2\beta_2)w = \alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2}$$

Likewise, in the second equation, $p$ has a component $u_w$.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\text{OLS}} = \frac{\sum (w_i - \bar{w})(p_i - \bar{p})}{\sum (w_i - \bar{w})^2}$$

We will investigate the sign of the bias in the slope coefficient if OLS is used to fit the price inflation equation.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\text{OLS}} = \frac{\sum (w_i - \bar{w})(p_i - \bar{p})}{\sum (w_i - \bar{w})^2}$$

reduced form equation

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

As usual we start by substituting for the dependent variable using the true model. For this purpose, we could use either the structural equation or the reduced form equation for *p*.

# SIMULTANEOUS EQUATIONS BIAS

**structural equations**

$$p = \beta_1 + \beta_2 w + u_p$$

$$w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\text{OLS}} = \frac{\sum (w_i - \bar{w})(p_i - \bar{p})}{\sum (w_i - \bar{w})^2}$$

$$= \frac{\sum (w_i - \bar{w})\left([\beta_1 + \beta_2 w_i + u_{pi}] - [\beta_1 + \beta_2 \bar{w} + \bar{u}_p]\right)}{\sum (w_i - \bar{w})^2}$$

**reduced form equation**

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

The algebra is simpler if we use the structural equation.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\text{OLS}} = \frac{\sum (w_i - \overline{w})(p_i - \overline{p})}{\sum (w_i - \overline{w})^2}$$

$$= \frac{\sum (w_i - \overline{w})\left([\beta_1 + \beta_2 w_i + u_{pi}] - [\beta_1 + \beta_2 \overline{w} + \overline{u}_p]\right)}{\sum (w_i - \overline{w})^2}$$

$$= \frac{\sum (w_i - \overline{w})\left(\beta_2 [w_i - \overline{w}] + u_{pi} - \overline{u}_p\right)}{\sum (w_i - \overline{w})^2}$$

The $\beta_1$ terms cancel. We rearrange the rest of the second factor in the numerator.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\begin{aligned}
b_2^{\mathrm{OLS}} &= \frac{\sum (w_i - \overline{w})(p_i - \overline{p})}{\sum (w_i - \overline{w})^2} \\
&= \frac{\sum (w_i - \overline{w})\left([\beta_1 + \beta_2 w_i + u_{pi}] - [\beta_1 + \beta_2 \overline{w} + \overline{u}_p]\right)}{\sum (w_i - \overline{w})^2} \\
&= \frac{\sum (w_i - \overline{w})\left(\beta_2 [w_i - \overline{w}] + u_{pi} - \overline{u}_p\right)}{\sum (w_i - \overline{w})^2} \\
&= \beta_2 + \frac{\sum (w_i - \overline{w})\left(u_{pi} - \overline{u}_p\right)}{\sum (w_i - \overline{w})^2}
\end{aligned}$$

Hence we obtain the usual decomposition into true value and error term.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\mathrm{OLS}} = \beta_2 + \frac{\sum (w_i - \overline{w})(u_{pi} - \overline{u}_p)}{\sum (w_i - \overline{w})^2}$$

We will now investigate the properties of the error term. Of course, we would like it to have expected value 0, making the estimator unbiased.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\text{OLS}} = \beta_2 + \frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2}$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2}$$

However, the error term is a nonlinear function of both $u_p$ and $u_w$ because both are components of $w$. As a consequence, it is not possible to obtain a closed-form analytical expression for its expected value.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\text{OLS}} = \beta_2 + \frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2}$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2}$$

We will investigate the large-sample properties instead. We will demonstrate that the estimator is inconsistent, and this will imply that it has undesirable finite-sample properties.

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\mathrm{OLS}} = \beta_2 + \frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2}$$

$$\mathrm{plim}\left( \frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2} \right)$$

$$\mathrm{plim}\left(\frac{A}{B}\right) = \frac{\mathrm{plim}\, A}{\mathrm{plim}\, B}$$

if $A$ and $B$ have probability limits and plim $B$ is not 0.

We focus on the error term. We would like to use the plim quotient rule. The plim of a quotient is the plim of the numerator divided by the plim of the denominator, provided that both of these limits exist.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\text{OLS}} = \beta_2 + \frac{\sum (w_i - \overline{w})(u_{pi} - \overline{u}_p)}{\sum (w_i - \overline{w})^2}$$

$$\operatorname{plim}\left( \frac{\sum (w_i - \overline{w})(u_{pi} - \overline{u}_p)}{\sum (w_i - \overline{w})^2} \right)$$

$$\operatorname{plim}\left(\frac{A}{B}\right) = \frac{\operatorname{plim} A}{\operatorname{plim} B}$$

if $A$ and $B$ have probability limits and plim $B$ is not 0.

However, as the expression stands, the numerator and the denominator of the error term do not have limits. The denominator increases indefinitely as the sample size increases. The nominator has no particular limit.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\mathrm{OLS}} = \beta_2 + \frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2}$$

$$\operatorname{plim}\left(\frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2}\right) = \operatorname{plim}\left(\frac{\frac{1}{n}\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\frac{1}{n}\sum (w_i - \bar{w})^2}\right)$$

$$\operatorname{plim}\left(\frac{A}{B}\right) = \frac{\operatorname{plim} A}{\operatorname{plim} B}$$

if *A* and *B* have probability limits and plim *B* is not 0.

To deal with this problem, we divide both the numerator and the denominator by *n*.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\mathrm{OLS}} = \beta_2 + \frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2}$$

$$\operatorname{plim}\left( \frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2} \right) = \operatorname{plim}\left( \frac{\frac{1}{n}\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\frac{1}{n}\sum (w_i - \bar{w})^2} \right)$$

$$\operatorname{plim} \frac{1}{n}\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p) = \operatorname{cov}(w, u_p)$$

$$\operatorname{plim} \frac{1}{n}\sum (w_i - \bar{w})^2 = \operatorname{var}(w)$$

It can be shown that the limit of the numerator is the covariance of $w$ and $u_p$ and the limit of the denominator is the variance of $w$.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$b_2^{\text{OLS}} = \beta_2 + \frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2}$$

$$\text{plim}\left( \frac{\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\sum (w_i - \bar{w})^2} \right) = \text{plim}\left( \frac{\frac{1}{n}\sum (w_i - \bar{w})(u_{pi} - \bar{u}_p)}{\frac{1}{n}\sum (w_i - \bar{w})^2} \right)$$

$$= \frac{\text{cov}(w, u_p)}{\text{var}(w)}$$

Hence the numerator and the denominator of the error term have limits and we are entitled to implement the plim quotient rule. We need var($w$) to be non-zero, but this will be the case assuming that there is some variation in $w$.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\operatorname{plim} b_2^{\mathrm{OLS}} = \beta_2 + \frac{\operatorname{cov}(u_p, w)}{\operatorname{var}(w)}$$

$$\operatorname{cov}(u_p, w) = \operatorname{cov}\left( u_p, \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2} \right)$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2}$$

We will now derive the limiting value of the numerator. The first step is to substitute for *w* from its reduced form equation. (*Note*: Here we must use the reduced form equation. If we use the structural equation, we will find ourselves going round in circles.)

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\text{plim}\, b_2^{\text{OLS}} = \beta_2 + \frac{\text{cov}(u_p, w)}{\text{var}(w)}$$

$$\text{cov}(u_p, w) = \text{cov}\left( u_p, \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2} \right)$$

$$= \frac{1}{1 - \alpha_2\beta_2} \left\{ \begin{array}{l} \text{cov}(u_p, [\alpha_1 + \alpha_2\beta_1]) + \text{cov}(u_p, \alpha_3 U) \\ + \text{cov}(u_p, \alpha_2 u_p) + \text{cov}(u_p, u_w) \end{array} \right\}$$

We use Covariance Rule 1 to decompose the expression.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\operatorname{plim} b_2^{\text{OLS}} = \beta_2 + \frac{\operatorname{cov}(u_p, w)}{\operatorname{var}(w)}$$

$$\operatorname{cov}(u_p, w) = \operatorname{cov}\left( u_p, \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2} \right)$$

$$= \frac{1}{1 - \alpha_2\beta_2} \left\{ \begin{array}{l} \operatorname{cov}(u_p, [\alpha_1 + \alpha_2\beta_1]) + \operatorname{cov}(u_p, \alpha_3 U) \\ + \operatorname{cov}(u_p, \alpha_2 u_p) + \operatorname{cov}(u_p, u_w) \end{array} \right\}$$

$$= \frac{1}{1 - \alpha_2\beta_2} \left( 0 + 0 + \alpha_2 \operatorname{var}(u_p) + 0 \right) = \frac{\alpha_2 \sigma_{u_p}^2}{1 - \alpha_2\beta_2}$$

The first term is 0 because ($\alpha_1 + \alpha_2\beta_1$) is a constant. The second term is 0 because *U* is exogenous and so distributed independently of $u_p$

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\operatorname{plim} b_2^{\text{OLS}} = \beta_2 + \frac{\operatorname{cov}(u_p, w)}{\operatorname{var}(w)}$$

$$\operatorname{cov}(u_p, w) = \operatorname{cov}\left( u_p, \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2} \right)$$

$$= \frac{1}{1 - \alpha_2\beta_2} \left\{ \begin{array}{l} \operatorname{cov}(u_p, [\alpha_1 + \alpha_2\beta_1]) + \operatorname{cov}(u_p, \alpha_3 U) \\ + \operatorname{cov}(u_p, \alpha_2 u_p) + \operatorname{cov}(u_p, u_w) \end{array} \right\}$$

$$= \frac{1}{1 - \alpha_2\beta_2} \left( 0 + 0 + \alpha_2 \operatorname{var}(u_p) + 0 \right) = \frac{\alpha_2 \sigma_{u_p}^2}{1 - \alpha_2\beta_2}$$

The fourth term is 0 if the disturbance terms are distributed independently of each other. This is not necessarily the case but, for simplicity, we will assume it to be true.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\operatorname{plim} b_2^{\mathrm{OLS}} = \beta_2 + \frac{\operatorname{cov}(u_p, w)}{\operatorname{var}(w)}$$

$$\operatorname{cov}(u_p, w) = \operatorname{cov}\left( u_p, \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2} \right)$$

$$= \frac{1}{1 - \alpha_2\beta_2} \left\{ \begin{array}{l} \operatorname{cov}(u_p, [\alpha_1 + \alpha_2\beta_1]) + \operatorname{cov}(u_p, \alpha_3 U) \\ + \operatorname{cov}(u_p, \alpha_2 u_p) + \operatorname{cov}(u_p, u_w) \end{array} \right\}$$

$$= \frac{1}{1 - \alpha_2\beta_2} \left( 0 + 0 + \alpha_2 \operatorname{var}(u_p) + 0 \right) = \frac{\alpha_2 \sigma_{u_p}^2}{1 - \alpha_2\beta_2}$$

However, the third term is nonzero because the limiting value of a sample variance is the corresponding population variance.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\text{plim}\, b_2^{\text{OLS}} = \beta_2 + \frac{\text{cov}(u_p, w)}{\text{var}(w)}$$

$$\text{var}(w) = \text{var}\left(\frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2}\right)$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2}$$

**If we were interested in obtaining an explicit mathematical expression for the large-sample bias, we would decompose var(*w*) in the same way, substituting for *w* from the reduced form, expanding, and then simplifying as best we can.**

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\operatorname{plim} b_2^{\text{OLS}} = \beta_2 + \frac{\operatorname{cov}(u_p, w)}{\operatorname{var}(w)}$$

$$\operatorname{cov}(u_p, w) = \frac{\alpha_2 \sigma_{u_p}^2}{1 - \alpha_2 \beta_2}$$

However, usually we are content with determining the sign of the large sample bias, if we can. Since variances are always positive, the sign of the bias will depend on the sign of cov($u_p$,$w$).

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\operatorname{plim} b_2^{\mathrm{OLS}} = \beta_2 + \frac{\operatorname{cov}(u_p, w)}{\operatorname{var}(w)}$$

$$\operatorname{cov}(u_p, w) = \frac{\alpha_2 \sigma_{u_p}^2}{1 - \alpha_2 \beta_2}$$

The sign of the bias will depend on the sign of the term ($1 - \alpha_2\beta_2$), since $\alpha_2$ must be positive and the variance components are positive.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = \beta_1 + \beta_2 \left( \alpha_1 + \alpha_2 p + \alpha_3 U + u_w \right) + u_p$$

$$\left(1 - \alpha_2 \beta_2\right) p = \beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w$$

reduced form equation

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

$$w = \alpha_1 + \alpha_2 \left( \beta_1 + \beta_2 w + u_p \right) + \alpha_3 U + u_w$$

$$\left(1 - \alpha_2 \beta_2\right) w = \alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2}$$

**Looking at the reduced form equation for *w*, *w* should be a decreasing function of *U*. $\alpha_3$ should be negative. So $(1 - \alpha_2\beta_2)$ must be positive. We conclude that, in this particular case, the large-sample bias is positive.**

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\Delta w = \alpha_3 \Delta U$$

In fact, $(1 - \alpha_2\beta_2)$ being positive is a condition for the existence of equilibrium in this model. Suppose that the exogenous variable $U$ changed by an amount $\Delta U$. The immediate effect on $w$ would be to change it by $\alpha_3 \Delta U$ (in the opposite direction, since $\alpha_3 < 0$).

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\Delta w = \alpha_3 \Delta U$$

$$\Delta p = \beta_2 \Delta w = \beta_2 \alpha_3 \Delta U$$

This would cause *p* to change by $\beta_2 \alpha_3 \Delta U$.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\Delta w = \alpha_3 \Delta U$$

$$\Delta p = \beta_2 \Delta w = \beta_2 \alpha_3 \Delta U$$

$$\begin{aligned} \Delta w &= \alpha_3 \Delta U + \alpha_2 \Delta p \\ &= (1 + \alpha_2 \beta_2) \alpha_3 \Delta U \end{aligned}$$

**This would cause a secondary change in *w* equal to $\alpha_2 \beta_2 \alpha_3 \Delta U$.**

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\Delta w = \alpha_3 \Delta U$$

$$\Delta p = \beta_2 \Delta w = \beta_2 \alpha_3 \Delta U$$

$$\begin{aligned} \Delta w &= \alpha_3 \Delta U + \alpha_2 \Delta p \\ &= \left(1 + \alpha_2 \beta_2\right) \alpha_3 \Delta U \end{aligned}$$

$$\Delta p = \beta_2 \Delta w = \beta_2 \left(1 + \alpha_2 \beta_2\right) \alpha_3 \Delta U$$

This in turn would cause *p* to change by a secondary amount $\alpha_2 \beta_2^2 \alpha_3 \Delta U$.

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\Delta w = \alpha_3 \Delta U$$

$$\Delta p = \beta_2 \Delta w = \beta_2 \alpha_3 \Delta U$$

$$\begin{aligned} \Delta w &= \alpha_3 \Delta U + \alpha_2 \Delta p \\ &= (1 + \alpha_2 \beta_2) \alpha_3 \Delta U \end{aligned}$$

$$\Delta p = \beta_2 \Delta w = \beta_2 (1 + \alpha_2 \beta_2) \alpha_3 \Delta U$$

$$\begin{aligned} \Delta w &= \alpha_3 \Delta U + \alpha_2 \Delta p \\ &= (1 + \alpha_2 \beta_2 + \alpha_2^2 \beta_2^2) \alpha_3 \Delta U \end{aligned}$$

**This would cause *w* to change by a further amount $\alpha_2^2 \beta_2^2 \alpha_3 \Delta U$.**

# SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\Delta w = \alpha_3 \Delta U$$

$$\Delta p = \beta_2 \Delta w = \beta_2 \alpha_3 \Delta U$$

$$\begin{aligned} \Delta w &= \alpha_3 \Delta U + \alpha_2 \Delta p \\ &= (1 + \alpha_2 \beta_2) \alpha_3 \Delta U \end{aligned}$$

$$\Delta p = \beta_2 \Delta w = \beta_2 (1 + \alpha_2 \beta_2) \alpha_3 \Delta U$$

$$\begin{aligned} \Delta w &= \alpha_3 \Delta U + \alpha_2 \Delta p \\ &= (1 + \alpha_2 \beta_2 + \alpha_2^2 \beta_2^2) \alpha_3 \Delta U \\ \alpha_2 \beta_2 < 1 \qquad &= (1 + \alpha_2 \beta_2 + \alpha_2^2 \beta_2^2 + \alpha_2^3 \beta_2^3 + \ldots) \alpha_3 \Delta U \end{aligned}$$

And so on and so forth. The total change will be finite only if $\alpha_2\beta_2 < 1$. Otherwise the process would be explosive, which is implausible.

## SIMULTANEOUS EQUATIONS BIAS

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\text{plim}\, b_2^{\text{OLS}} = \beta_2 + \frac{\text{cov}(u_p, w)}{\text{var}(w)}$$

$$\text{cov}(u_p, w) = \frac{\alpha_2 \sigma_{u_p}^2}{1 - \alpha_2 \beta_2}$$

Either way, we have demonstrated that $1 - \alpha_2\beta_2 > 0$ and hence that, in this case, the bias is positive. Note that one cannot generalize about the direction of simultaneous equations bias. It depends on the structure of the model.

# SIMULTANEOUS EQUATIONS BIAS: GRAPHICAL ILLUSTRATION & SIMULATION

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

In the previous slideshow we determined analytically the large-sample simultaneous equations bias for the price inflation / wage inflation model. Next we will look at the bias graphically.

# SIMULTANEOUS EQUATIONS BIAS: GRAPHICAL ILLUSTRATION & SIMULATION

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\mathrm{var}(w) = \mathrm{var}\left(\frac{\alpha_1 + \alpha_2\beta_1}{1-\alpha_2\beta_2} + \frac{\alpha_3 U + \alpha_2 u_p + u_w}{1-\alpha_2\beta_2}\right) = \mathrm{var}\left(\frac{\alpha_3 U + \alpha_2 u_p + u_w}{1-\alpha_2\beta_2}\right)$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1-\alpha_2\beta_2}$$

For this, it will be necessary to obtain an explicit expression for var(*w*). Variances are unaffected by additive constants, so the first part of the expression may be dropped.

# SIMULTANEOUS EQUATIONS BIAS: GRAPHICAL ILLUSTRATION & SIMULATION

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\mathrm{var}(w) = \mathrm{var}\left( \frac{\alpha_1 + \alpha_2\beta_1}{1-\alpha_2\beta_2} + \frac{\alpha_3 U + \alpha_2 u_p + u_w}{1-\alpha_2\beta_2} \right) = \mathrm{var}\left( \frac{\alpha_3 U + \alpha_2 u_p + u_w}{1-\alpha_2\beta_2} \right)$$

$$= \left\{ \frac{1}{(1-\alpha_2\beta_2)^2} \right\} \left\{ \begin{array}{l} \mathrm{var}(\alpha_3 U) + \mathrm{var}(\alpha_2 u_p) + \mathrm{var}(u_w) \\ + 2\,\mathrm{cov}(\alpha_3 U, u_w) + 2\,\mathrm{cov}(\alpha_3 U, \alpha_2 u_p) \\ + 2\,\mathrm{cov}(\alpha_2 u_p, u_w) \end{array} \right\}$$

Using Variance Rule 1, the numerator decomposes into three variances and three covariances.  The denominator is a constant common factor and may be taken, squared, outside the expression using Variance Rule 2.

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\mathrm{var}(w) = \mathrm{var}\left( \frac{\alpha_1 + \alpha_2 \beta_1}{1 - \alpha_2 \beta_2} + \frac{\alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2} \right) = \mathrm{var}\left( \frac{\alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2} \right)$$

$$= \left\{ \frac{1}{(1 - \alpha_2 \beta_2)^2} \right\} \left\{ \begin{array}{l} \mathrm{var}(\alpha_3 U) + \mathrm{var}(\alpha_2 u_p) + \mathrm{var}(u_w) \\ + 2\,\mathrm{cov}(\alpha_3 U, u_w) + 2\,\mathrm{cov}(\alpha_3 U, \alpha_2 u_p) \\ + 2\,\mathrm{cov}(\alpha_2 u_p, u_w) \end{array} \right\}$$

$$= \frac{\alpha_3^2 \sigma_U^2 + \alpha_2^2 \sigma_{u_p}^2 + \sigma_{u_w}^2}{(1 - \alpha_2 \beta_2)^2}$$

The covariances are all 0 on the assumption that $U$, $u_p$, and $u_w$ are distributed independently of each other. Thus the numerator consists of the variance expressions. Remember that we have to square $\alpha_3$ and $\alpha_2$ when we take them out of the variance expressions.

# SIMULTANEOUS EQUATIONS BIAS: GRAPHICAL ILLUSTRATION & SIMULATION

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\operatorname{plim} b_2^{\mathrm{OLS}} = \beta_2 + \frac{\operatorname{cov}(u_p, w)}{\operatorname{var}(w)}$$

$$\operatorname{cov}(u_p, w) = \frac{\alpha_2 \sigma_{u_p}^2}{1 - \alpha_2 \beta_2}$$

$$\operatorname{var}(w) = \frac{\alpha_3^2 \sigma_U^2 + \alpha_2^2 \sigma_{u_p}^2 + \sigma_{u_w}^2}{(1 - \alpha_2 \beta_2)^2}$$

$$\operatorname{plim} b_2^{\mathrm{OLS}} = \beta_2 + (1 - \alpha_2 \beta_2) \frac{\alpha_2 \sigma_{u_p}^2}{\alpha_3^2 \sigma_U^2 + \alpha_2^2 \sigma_{u_p}^2 + \sigma_{u_w}^2}$$

Hence we obtain an explicit expression for the plim of the slope coefficient.

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\operatorname{plim} b_2^{\mathrm{OLS}} = \beta_2 + (1 - \alpha_2\beta_2)\frac{\alpha_2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2}$$

$$= \beta_2 + \left(\frac{1}{\alpha_2} - \beta_2\right)\frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2}$$

To look at the bias graphically, it is helpful to rearrange the expression.

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\begin{aligned}
\operatorname{plim} b_2^{\mathrm{OLS}} &= \beta_2 + (1 - \alpha_2\beta_2)\frac{\alpha_2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2} \\
&= \beta_2 + \left(\frac{1}{\alpha_2} - \beta_2\right)\frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2} \\
&= \beta_2\left(1 - \frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2}\right) + \frac{1}{\alpha_2}\left(\frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2}\right)
\end{aligned}$$

We have now gathered the terms involving $\beta_2$ together.

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\begin{aligned}
\operatorname{plim} b_2^{\mathrm{OLS}} &= \beta_2 + (1 - \alpha_2\beta_2)\frac{\alpha_2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2} \\
&= \beta_2 + \left(\frac{1}{\alpha_2} - \beta_2\right)\frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2} \\
&= \beta_2\left(1 - \frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2}\right) + \frac{1}{\alpha_2}\left(\frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2}\right) \\
&= \beta_2\left(\frac{\alpha_3^2\sigma_U^2 + \sigma_{u_w}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2}\right) + \frac{1}{\alpha_2}\left(\frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2\sigma_U^2 + \alpha_2^2\sigma_{u_p}^2 + \sigma_{u_w}^2}\right)
\end{aligned}$$

Thus we see that the limiting value of the OLS estimator is a weighted average of the true value $\beta_2$ and $1/\alpha_2$.

# SIMULTANEOUS EQUATIONS BIAS: GRAPHICAL ILLUSTRATION & SIMULATION

structural equations

$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$\operatorname{plim} b_2^{\mathrm{OLS}} = \beta_2 \left( \frac{\alpha_3^2 \sigma_U^2 + \sigma_{u_w}^2}{\alpha_3^2 \sigma_U^2 + \alpha_2^2 \sigma_{u_p}^2 + \sigma_{u_w}^2} \right) + \frac{1}{\alpha_2} \left( \frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2 \sigma_U^2 + \alpha_2^2 \sigma_{u_p}^2 + \sigma_{u_w}^2} \right)$$

reduced form equation

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2}$$

Variations in *U* cause variations in *p* and *w*, the change in *p* being $\beta_2$ times the change in *w*. Such movements trace out the true relationship.

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reduced form equation

$$w = \frac{\alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2}$$

Likewise, variations in $u_p$ cause variations in $p$ and $w$, the change in $p$ being $1/\alpha_2$ times the change in $w$.

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$$p = \beta_1 + \beta_2 w + u_p \qquad w = \alpha_1 + \alpha_2 p + \alpha_3 U + u_w$$

$$p = 1.5 + 0.5w + u_p \qquad w = 2.5 + 0.5p - 0.4U$$

$n$ = 20.

$U$ = 2 to 6.75 in steps of 0.25.  Variance 2.08.

$u_p$ ~ iid $N$(0, 0.64).

$u_w$ dropped.

We will illustrate this with a Monte Carlo model, choosing parameter values as shown above.  The sample size 1s 20 and the values of $U$ are 2, to 6.75 in steps of 0.25.

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reduced form equation

$$p = \frac{\beta_1 + \alpha_1\beta_2 + \alpha_3\beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2\beta_2} = 3.67 - 0.27U + 1.33u_p$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2\beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2\beta_2} = 4.33 - 0.53U + 0.67u_p$$

Given the values of the parameters, the reduced form equations are as shown.

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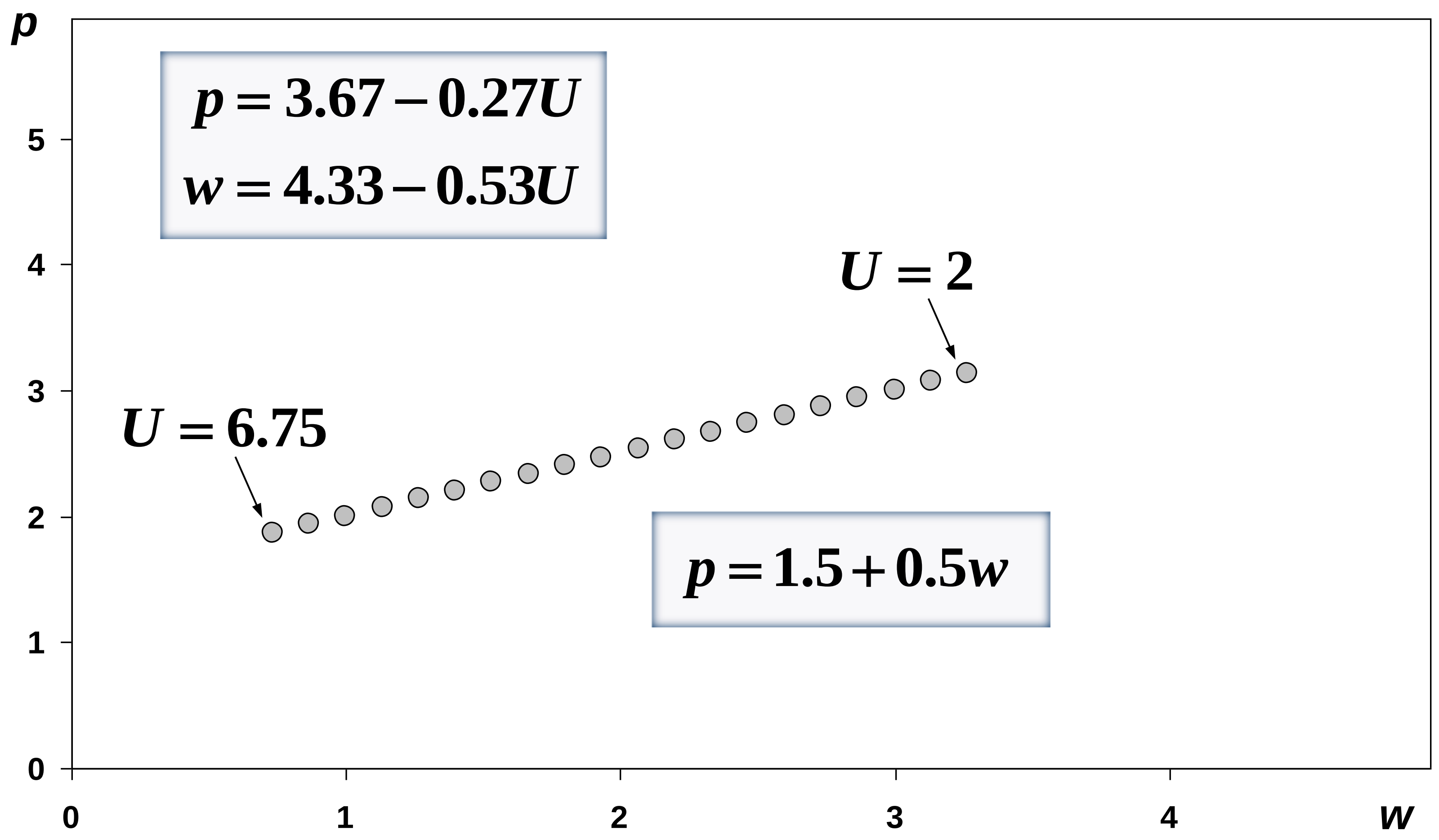
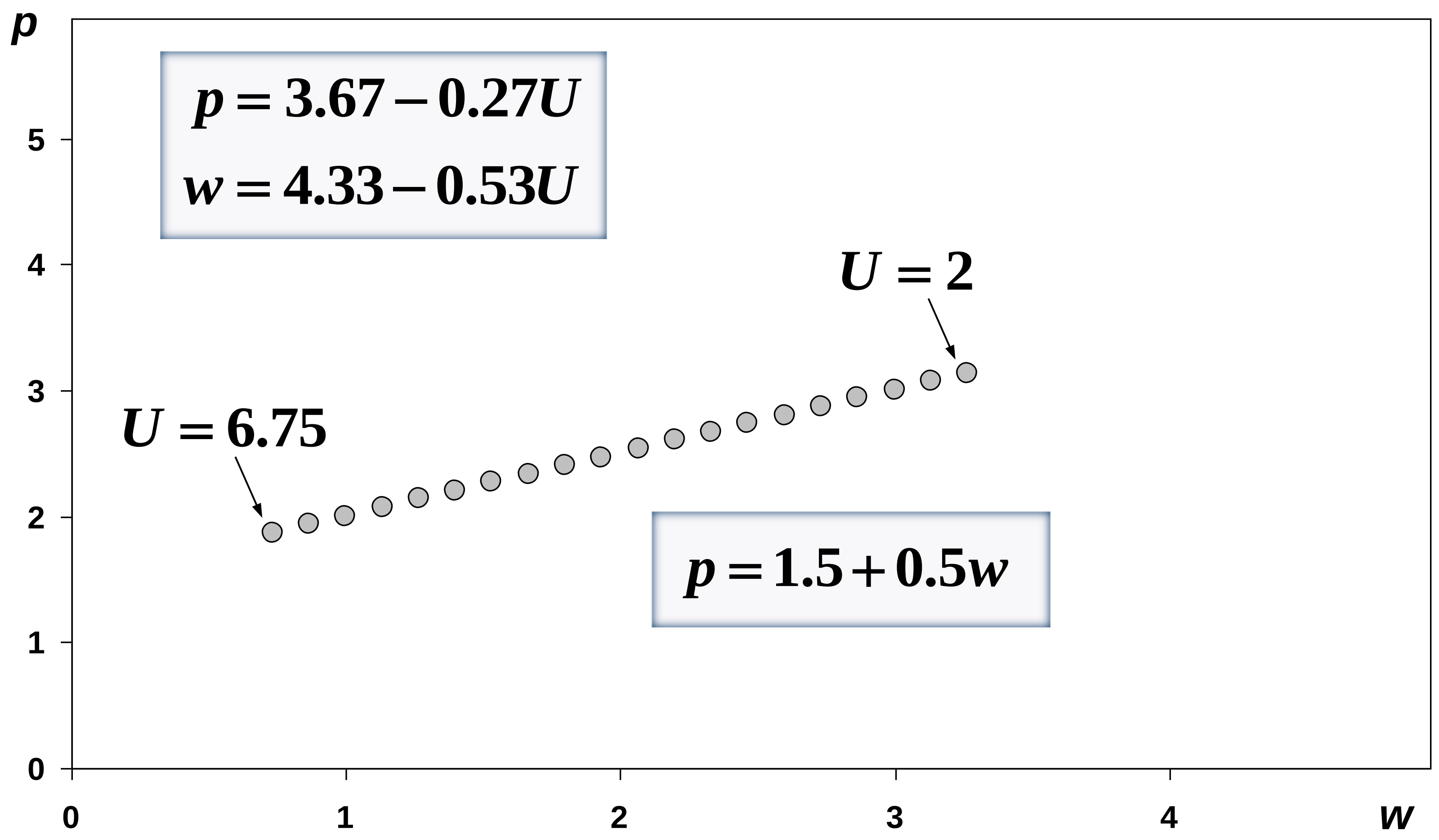



This is what we would see if there were no disturbance terms in the model.




**Note that the values of *p* and *w* in each observation are jointly determined by the value of *U* in that observation. A change in *U* changes *p* by only half the amount it changes *w*. Thus the slope is 0.5.**

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$$\operatorname{plim} b_2^{\text{OLS}} = \beta_2 \left( \frac{\alpha_3^2 \sigma_U^2 + \sigma_{u_w}^2}{\alpha_3^2 \sigma_U^2 + \alpha_2^2 \sigma_{u_p}^2 + \sigma_{u_w}^2} \right) + \frac{1}{\alpha_2} \left( \frac{\alpha_2^2 \sigma_{u_p}^2}{\alpha_3^2 \sigma_U^2 + \alpha_2^2 \sigma_{u_p}^2 + \sigma_{u_w}^2} \right)$$

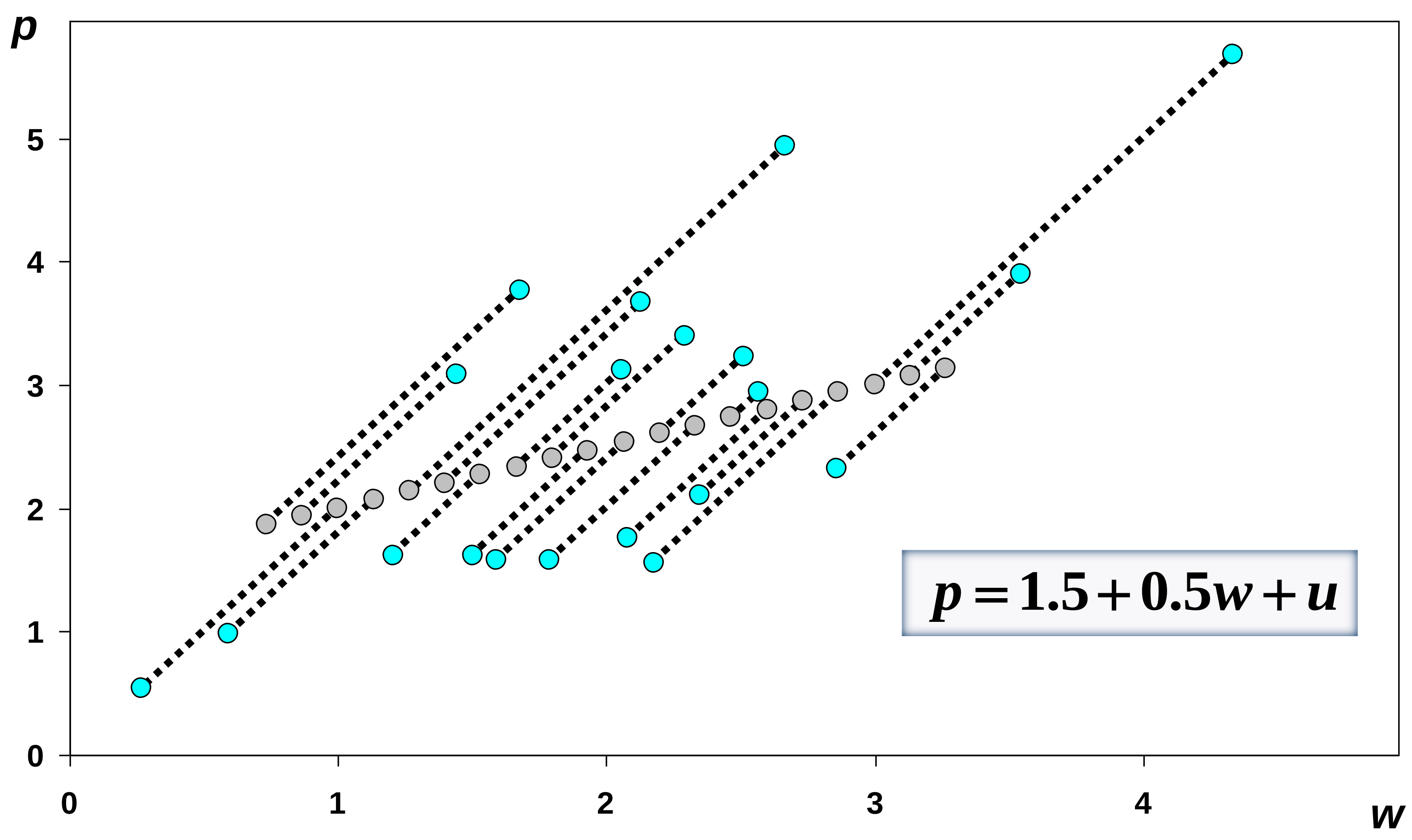
reduced form equation

$$p = \frac{\beta_1 + \alpha_1 \beta_2 + \alpha_3 \beta_2 U + u_p + \beta_2 u_w}{1 - \alpha_2 \beta_2}$$

reduced form equation

$$w = \frac{\alpha_1 + \alpha_2 \beta_1 + \alpha_3 U + \alpha_2 u_p + u_w}{1 - \alpha_2 \beta_2}$$

At the same time, *p* and *w* are both affected by variations in $u_p$, the change in *p* being $1/\alpha_2$ times the change in *w*.




As a consequence the actual observations are shifted away from the true relationship up or down along lines with slope $1/\alpha_2$, the dotted lines in the graph. Since $\alpha_2$ is 0.5, $1/\alpha_2$ is 2.0.

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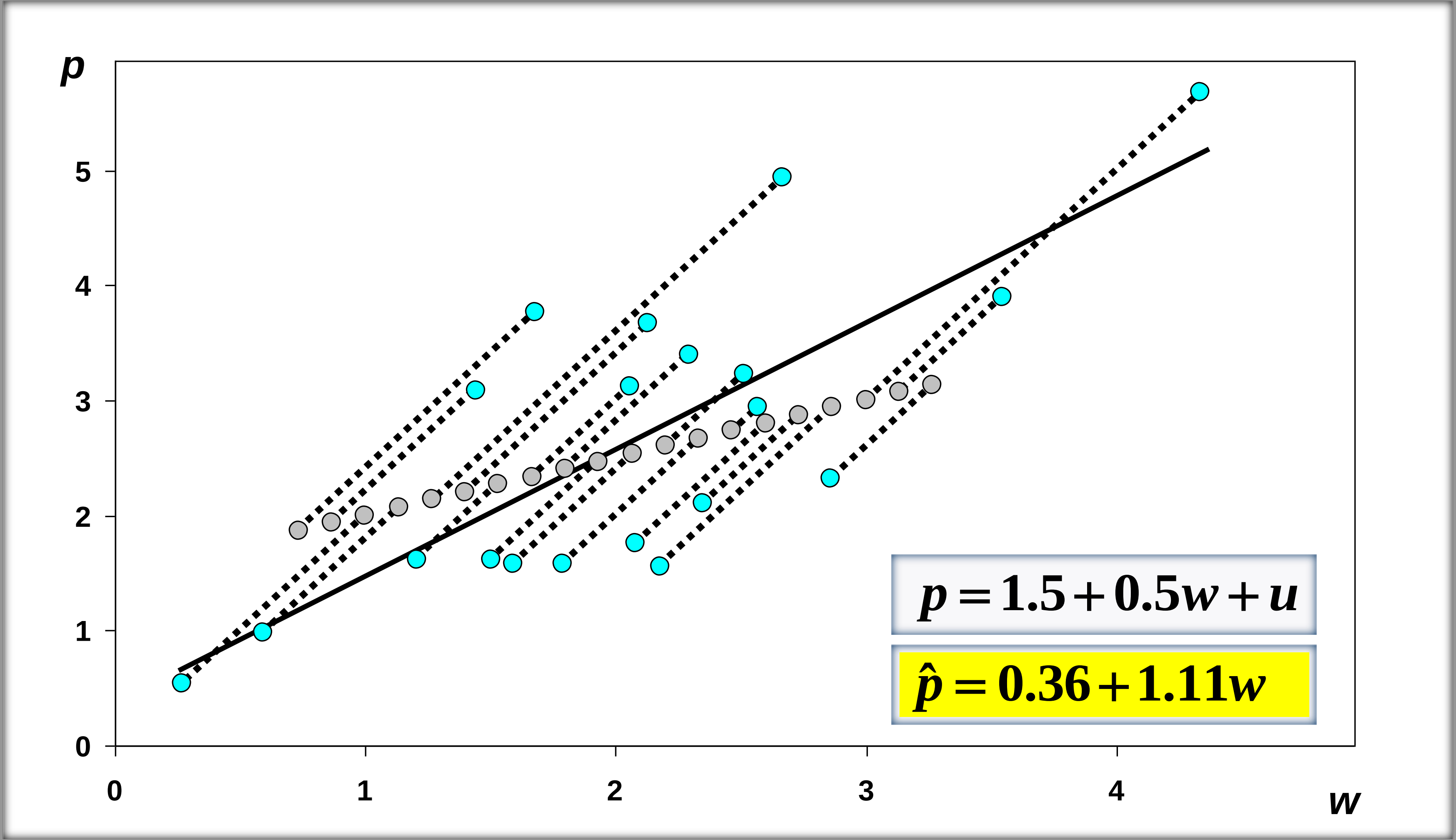


The regression line thus overestimates $\beta_2$.

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$$= 0.5 \left( \frac{0.16 \times 2.08}{0.16 \times 2.08 + 0.25 \times 0.64} \right)$$

$$+ 2.0 \left( \frac{0.25 \times 0.64}{0.16 \times 2.08 + 0.25 \times 0.64} \right)$$

$$\sigma_U^2 = 2.08$$

$$\sigma_{u_p}^2 = 0.64$$

$$\sigma_{u_w}^2 = 0$$

We will calculate the large-sample bias in the slope coefficient. *U* and $u_p$ were chosen so that they had population variances 2.08 and 0.64, respectively.

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$$+ 2.0 \left( \frac{0.25 \times 0.64}{0.16 \times 2.08 + 0.25 \times 0.64} \right)$$

$$= 0.99$$

$$\sigma_U^2 = 2.08$$

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$$\sigma_{u_w}^2 = 0$$

The large sample bias is 0.99. Our regression estimate, 1.11, was a little higher.

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| | $b_1$ | s.e.($b_1$) | $b_2$ | s.e.($b_2$) |
|---|---|---|---|---|
| 1 | 0.36 | 0.49 | 1.11 | 0.22 |
| 2 | 0.45 | 0.38 | 1.06 | 0.17 |
| 3 | 0.65 | 0.27 | 0.94 | 0.12 |
| 4 | 0.41 | 0.39 | 0.98 | 0.19 |
| 5 | 0.46 | 0.92 | 0.77 | 0.22 |
| 6 | 0.26 | 0.35 | 1.09 | 0.16 |
| 7 | 0.31 | 0.39 | 1.00 | 0.19 |
| 8 | 1.06 | 0.38 | 0.82 | 0.16 |
| 9 | –0.08 | 0.36 | 1.16 | 0.18 |
| 10 | 1.12 | 0.43 | 0.69 | 0.20 |

Here are the results for 10 samples in this simulation. The slope coefficient was overestimated every time and does appear to be distributed around its plim, 0.99.

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Because the slope coefficient was overestimated, the intercept was underestimated every time.

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No attention should be paid to the standard errors because they are invalidated by the simultaneous equations bias.

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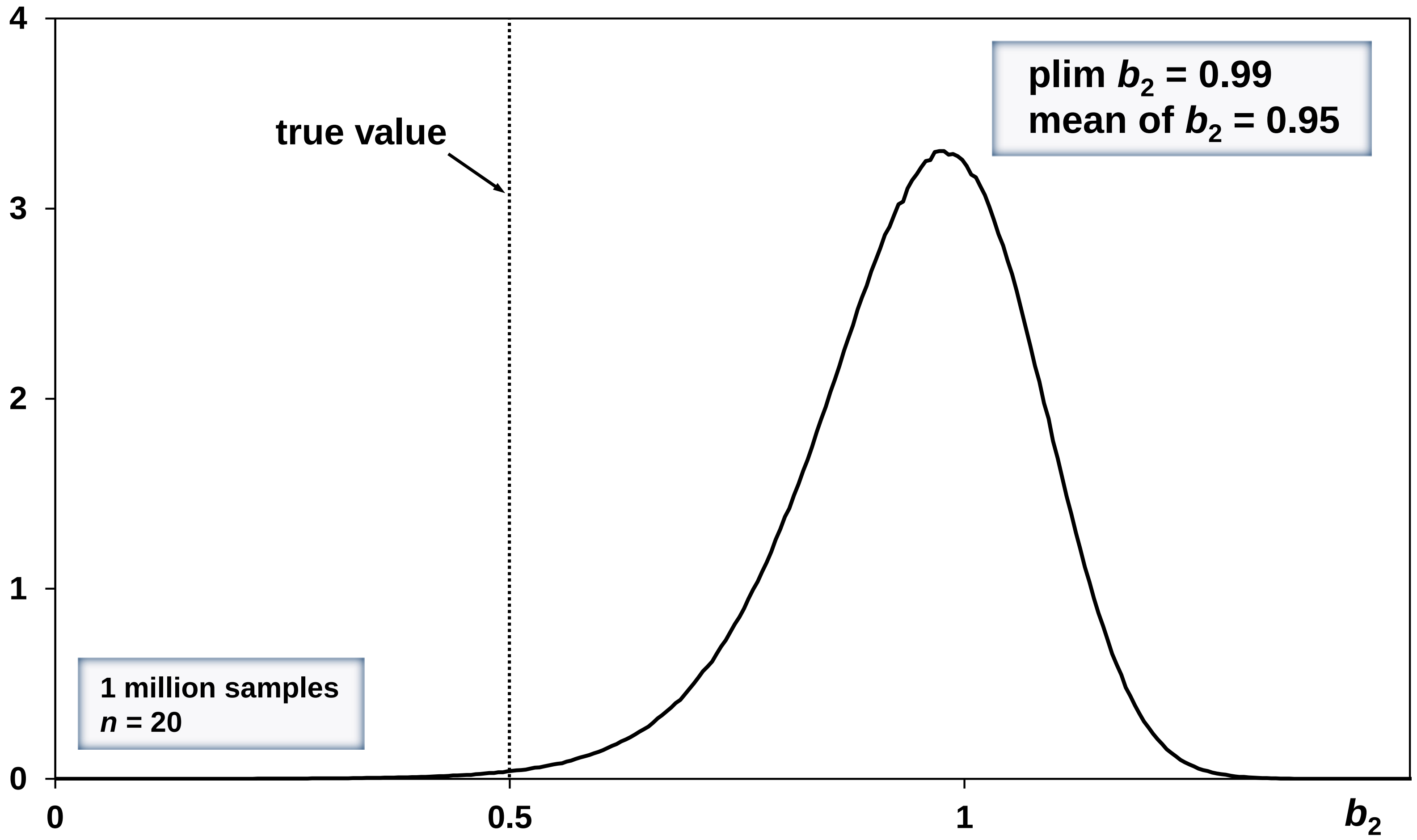


The chart plots the distribution of the slope coefficient for 1 million samples. Almost all the estimates are above the true value of 0.5, confirming the large-sample analysis.

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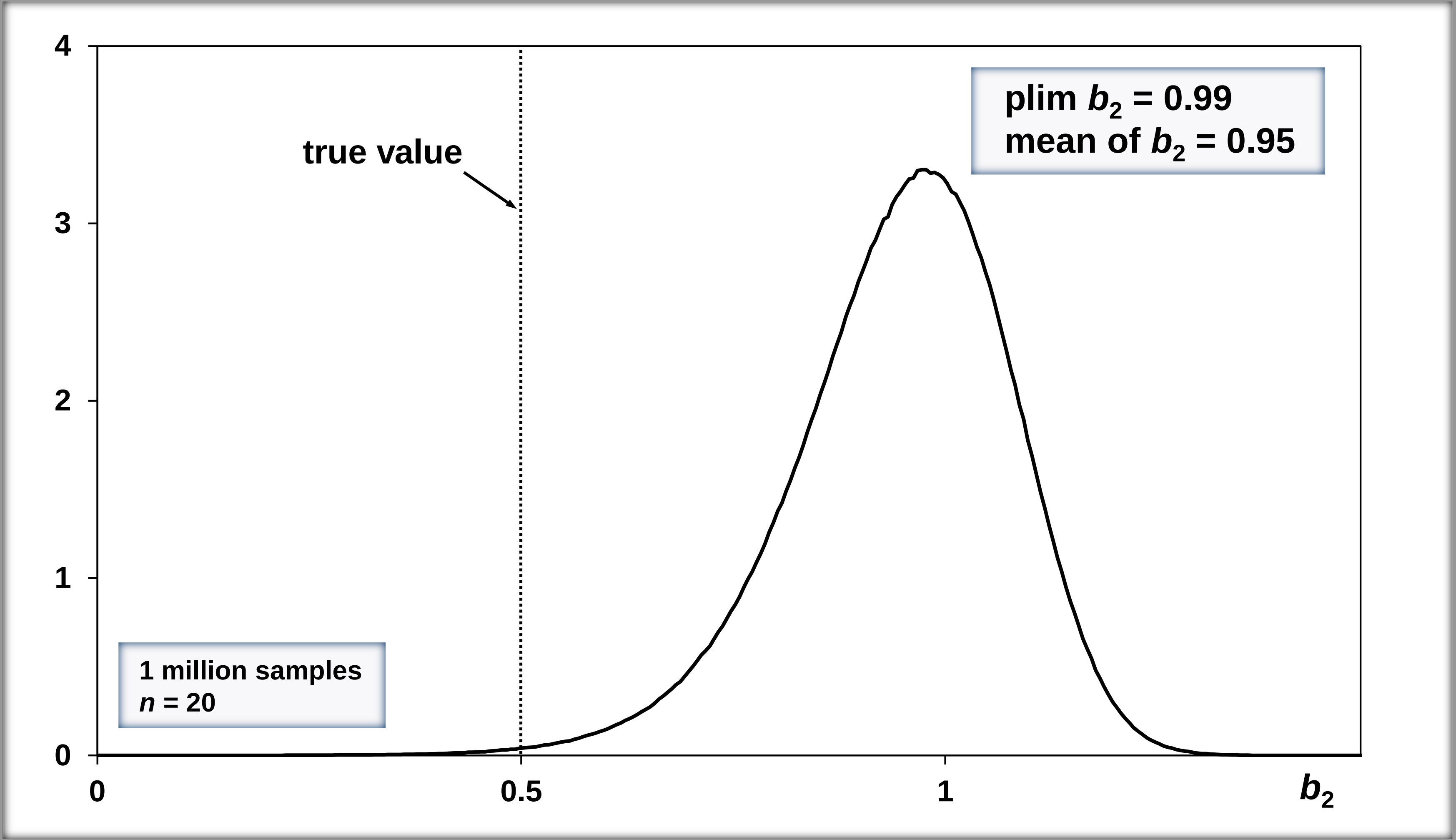


The plim (0.99) in this case provides a good guide to the size of the bias since the mean of the distribution is 0.95, fairly close to the plim even though, with only 20 observations, each sample was quite small.

The content of this slideshow comes from Section 9.2 of C. Dougherty, *Introduction to Econometrics*, fourth edition 2011, Oxford University Press. Additional (free) resources for both students and instructors may be downloaded from the OUP Online Resource Centre http://www.oup.com/uk/orc/bin/9780199567089/.

Individuals studying econometrics on their own who feel that they might benefit from participation in a formal course should consider the London School of Economics summer school course
EC212 Introduction to Econometrics
http://www2.lse.ac.uk/study/summerSchools/summerSchool/Home.aspx
or the University of London International Programmes distance learning course
EC2020 Elements of Econometrics
www.londoninternational.ac.uk/lse.

# Q & A

# Thank you very much!
# Vielen Dank!

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